Lecture 34

Lie triple system : $\mathfrak{m} \subset \mathfrak{g}$ subspace s.t. $x, y, z \in \mathfrak{m} \Rightarrow [[x,y],z] \in \mathfrak{m}$

Does not imply $\mathfrak{m}$ is a Lie alg.

e.g. Cartan decomp $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{p}$ is a Lie triple system.

Thm. $X = G/K$ sym space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then if $\mathfrak{m} \subset \mathfrak{g}$ is a Lie triple system and $\mathfrak{m}$ is a subspace of $\mathfrak{p}$, then

$$S = \exp(\mathfrak{m}) \cdot x_0$$

is a complete totally geod submfld of X. Every such submfld through x_0 has this form. All such S are themselves symmetric spaces.

Sketch. Given such $\mathfrak{m}$. Let $\mathfrak{k}' = [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

Then $\mathfrak{k}' \oplus \mathfrak{m}$ is a subalg of $\mathfrak{g}$:

$$[\mathfrak{k}', \mathfrak{m}] = [[\mathfrak{m},\mathfrak{m}],\mathfrak{m}] \subset \mathfrak{m}. \quad \text{(triple sys)}$$

$$[\mathfrak{k}', \mathfrak{k}'] = [[\mathfrak{m},\mathfrak{m}],[\mathfrak{m},\mathfrak{m}]] \overset{\text{jacobi}}{\subset} [[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}],\mathfrak{m}] \subset [\mathfrak{m},\mathfrak{m}]$$

$$[\mathfrak{m},\mathfrak{m}] = \mathfrak{k}' \quad \checkmark$$

Let G' = Lie sub w/ alg $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}$. $K' \subset G'$.

One shows G'/K' symm, sub of G/K.

Geodesics in G'/K' all like $\exp(tx)$, $x \in \mathfrak{m}$.

Those are geod in G/K as well. $\square$

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra.

(e.g. if G cplx, K the cpt real form assoc to G, H.

let $\mathfrak{a} \subset \mathfrak{h}$ be the subspace where roots are real.)

Such a subspace is called a Cartan subspace.

Thm. Cartan subspaces of $\mathfrak{p}$ lie in a single Ad_K-orbit.

Thm. $\exp(\mathfrak{a})\cdot x_0$ is isometric to $\mathbb{E}^r$ where $r = \dim \mathfrak{a}$.
These are exactly the maximal complete embedded flat tot geod submflds of $X = G/K$.

Pf. Abelian $\Rightarrow$ triple system, and $R = 0$. $\square$

Ex. $SL_n\mathbb{R}/SO(n)$. $\mathfrak{k} = X + X^t = 0$, $\mathfrak{p} = X - X^t = 0$. $x_0 = SO(n)$. or std inner.

$\mathfrak{a}$ = diagonal matrices.

$\exp(\mathfrak{a})\cdot x_0$ = inner prod of form $\langle v,v\rangle = \sum_i d_i v_i^2$ $\quad \prod d_i = 1$.
= inner prod s.t. $e_1 \dots e_n$ is an orthonorm basis

Other Cartan subspaces: Write $\mathbb{R}^n = L_1 \oplus \dots \oplus L_n$ with $L_i \perp L_j$. Let A be the grp of det 1 matrices preserving $L_1 \dots L_n$. Then use $\mathfrak{a} = \mathrm{Lie}(A)$.

$SO(n)$ acts transitively on Cartan subsp: rotate L_i to $\mathrm{span}(e_i)$.

Ex. $SL_n\mathbb{C}/SU(n)$. Same but $\mathfrak{a}$ will be diagonal real matrices.

Thm Every $X \in \mathfrak{p}$ lies in a Cartan subspace.
Thus, every geodesic of G/K lies in a flat and $\forall\ x, y \in G/K$ $\exists$ flat containing x and y.

Preview. $SL_n\mathbb{R}$ case. $T \in SL_n\mathbb{R}$. The singular value decomp says $T = UDV$ where $U, V \in SO(n)$, D diagonal.

$\Rightarrow T\,SO(n) = UDV\,SO(n) = UD\,SO(n) = UDU^{-1}\,SO(n)$

$\mathrm{Lie}(UDU^{-1}) = \mathrm{Ad}_U(\mathfrak{a})$ is the Cartan.

Here we also see that the containing Cartan is typically unique.

If D has repeated entries, then we can change U, V by rotations in those subspaces.

What's canonical is $T \neq id \rightsquigarrow$ decomp $\mathbb{R}^n = \bigoplus E_i$ and then any subordinate line decomp is a flat.

Actual argument: Same but $G = KAK$.

Visual boundary. Hadamard mfld = complete, simp connected, nonpositively curved.

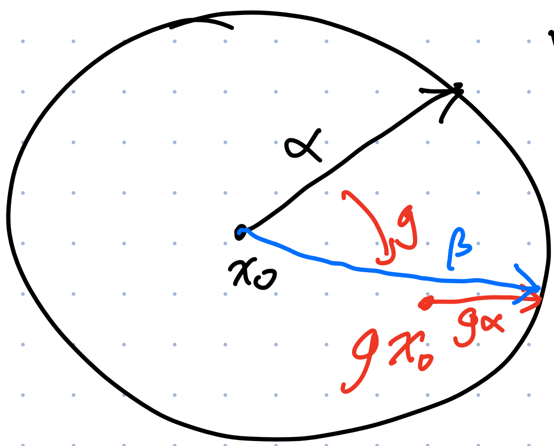
$\Rightarrow$ two geodesic rays from $x_0 \in M$ with distinct directions have $d(\gamma_1(t), \gamma_2(t)) \geq Ct$

$\Rightarrow \{\text{geod rays from } x_0\}/\sim \;\cong$ unit sphere in $T_{x_0}M$

In fact $\{\text{geod rays in } M\}/\sim \;\cong$ unit sphere in $T_x M$ for any $x \in M$.

This is called the visual boundary $\partial_{vis} G/K$

G acts on $\partial_{vis} G/K$, but the action is subtle.



$\forall \alpha \in T_{x_0} G/K$, $g \in G$ $\exists! \beta \in T_{x_0}$ s.t. $\beta \sim g \cdot \alpha$ in ∂_{vis}.

In general there are many G-orbits.

Thm. Each G-orbit in the vis bdry is a flag variety.